

current harmonics appear in plasmas subjected to an a.c. field. These authors made the following ansatz for the isotropic part f^0 :

$$f(v, t) = f_0^0(v) + f_2^0(v) \exp(2i\omega t) + v_x [f_1^1(v) \exp(i\omega t) + f_3^1(v) \exp(3i\omega t)].$$

In recent time similar series expansions have been applied in order to calculate the amplitudes of higher harmonics^{19–22} even if a *constant* magnetic field is taken into account^{23, 24}. The essential result of all these papers following the treatment of Margenau and Hartmann can be derived as follows:

Suppose the isotropic part f^0 to be time-dependent. By expanding f^0 into a Fourier series it is possible to obtain an infinite number of current harmonics in general. If in addition a constant mag-

netic field is taken into account this will modify the amplitudes of the higher harmonics without changing the time-behaviour, however, which is exclusively determined by the alternating electric field itself.

On the contrary the field configuration investigated in this paper (a.c. and d.c. field, rotating magnetic field, induced electric field) gives rise to a finite number of “higher magnetic cross harmonics” in the plasma under the condition that f^0 is time-independent. In another paper¹⁷ it is planned to give an explicit calculation of the isotropic part f^0 for the field configuration here discussed making use of the additional assumption that certain relations of homogeneity^{14, 15} are valid.

¹⁹ P. ROSEN, Phys. Fluids **4**, 341 [1961].

²⁰ J. KRENZ, Phys. Fluids **8**, 1871 [1965].

²¹ TING-WEI TANG, Phys. Fluids **9**, 415 [1966].

²² T. MORRONE, Phys. Fluids **10**, 1507 [1967].

²³ K. CHIYODA, J. Phys. Soc. Japan **20**, 290 [1965].

²⁴ M. S. SODHA, J. Phys. Soc. Japan **21**, 2674 [1966].

Generalized Electron Cyclotron Resonance in Weakly Ionized Time-Varying Magnetoplasmas

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The electron distribution function is calculated explicitly for a weakly ionized plasma under the action of an alternating electric field $\mathbf{E} = \{0, 0, E_0 \cos \omega t\}$ and a circularly polarized magnetic field $\mathbf{B}^R = B_0 \{\cos \omega_B t, \sin \omega_B t, 0\}$ rotating perpendicular to the a. c. field. Furthermore, a constant magnetic field $\mathbf{B}_0 = \{0, 0, B_0\}$ is taken into account. The isotropic part f^0 of the electron distribution function which contains, in special cases, well-known standard distributions (distributions of DRUYVENSTEYN, DAYDOV, MARGENAU, ALLIS, FAIN, GUREVIČ) shows a resonance behaviour if the frequencies ω , $\omega_c = (q/m) B_0$, $\omega_0 = (q/m) B_0$, and ω_B satisfy the relation

$$\omega = \sqrt{\omega_c^2 + (\omega_0 + \omega_B)^2}.$$

This can be understood as a generalized cyclotron resonance phenomenon.

A general treatment of weakly ionized time-varying magnetoplasmas by means of the kinetic theory (Boltzmann equation) was given in the paper¹. The application to a special field configuration consisting of an alternating electric field and a circularly polarized magnetic field was described in a following paper². Assuming the isotropic part f^0 of the electron distribution function to be time-independent one can determine the direction-dependent part f^1 in terms of the unknown part f^0 . This leads to higher

harmonic terms in the solution $f^1\{f^0\}$ which can be explained by a cross drift mechanism. It is worth noting that an explanation of these terms does not require the explicit knowledge of the quantity f^0 .

In this paper we are interested in a precise determination of f^0 in order to calculate f^1 (and also f as a whole) in its explicit form. This will enable us to investigate a special cyclotron resonance phenomenon appearing when the characteristic frequencies of the external fields satisfy a simple relation.

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¹ W. STILLER and G. VOJTA, Physica (to appear 1969).

² W. STILLER and G. VOJTA, Z. Naturforsch. **24 a**, 545 [1969].



If an electric d.c. field instead of an a.c. field is present it is furthermore possible to get the explicit form of f^0 considering the limit $\omega \rightarrow 0$. Finally it can be proved that the expression for f^0 goes over into well-known standard distributions if the corresponding simple field configurations are taken into account.

1. Calculation of the Isotropic Part f^0

Let us assume a weakly ionized plasma consisting of heavy particles (mass: M , temperature: T) which have a Maxwellian distribution and electrons (mass: m , charge: q) the velocity distribution of which we want to calculate. We consider the following field configurations:

- (i) Alternating electric field
 $\mathbf{E} = \{0, 0, E_{0z} \cos \omega t\}$;
- (ii) Rotating magnetic field
 $\mathbf{B}^R = B_C \{\cos \omega_B t, \sin \omega_B t, 0\}$,
 $\omega_C = (q/m) B_C$;
- (iii) Constant magnetic field
 $\mathbf{B}_0 = \{0, 0, B_0\}$, $\omega_0 = q B_0/m$;
- (iv) Induced electric field
 $\mathbf{E}_1^R = B_C \omega_B \{0, 0, x \cos \omega_B t + y \sin \omega_B t\}$.

Further we assume that

- (a) the following relations of homogeneity are valid

$$\partial f^0 / \partial \mathbf{r} = 0, \quad |\mathbf{E}_{1z}^R| \ll |\mathbf{E}_z|,$$

- (b) f^0 is time-independent.

The isotropic part f^0 of the electron distribution function can be calculated from the following differential equation^{1, 2}

$$\left. \begin{aligned} \frac{1}{3 v^2} \frac{\partial}{\partial v} (v^2 A_z \cos \omega t f_z^1(f^0)) &= \frac{\delta f^0}{\delta t} \Big|_{e-n}, \\ \frac{\delta f^0}{\delta t} \Big|_{e-n} &= \frac{1}{v^2} \frac{m}{M} \frac{\partial}{\partial v} \left(v^3 \nu f^0 + \frac{k T}{m} v^2 \nu \frac{\partial f^0}{\partial v} \right). \end{aligned} \right\} \quad (1)$$

In order to solve this equation it is, first of all, necessary to calculate the quantity $f_z^1(f^0)$ using the assumptions (a) and (b) and the fields given by (i) – (iv). With the help of the Tables 1 and 2 of paper² one finds easily

$$f_z^1(f^0) = \frac{A_z}{D} \frac{\partial f^0}{\partial v} [(\omega_{a11} \omega_{b11} + \omega_{a12} \omega_{b12}) \sin \omega t + (\omega_{a11} \omega_{b12} - \omega_{a12} \omega_{b11}) \cos \omega t] \quad (2)$$

where

$$\left. \begin{aligned} D &= (\omega_{a11})^2 + (\omega_{a12})^2, \quad A_z = (q/m) |\mathbf{E}_{0z}|, \\ \omega_{a11} &= \nu(v) [\nu^2(v) + \omega_c^2 + (\omega_0 + \omega_B)^2 - 3 \omega^2], \\ \omega_{a12} &= \omega [3 \nu^2(v) + \omega_c^2 + (\omega_0 + \omega_B)^2 - \omega^2], \\ \omega_{b11} &= 2 \nu(v) \omega, \\ \omega_{b12} &= -[\nu^2(v) + (\omega_0 + \omega_B)^2 - \omega^2]. \end{aligned} \right\} \quad (3)$$

Substituting $f_z^1(f^0)$ into Eq. (1) and averaging over a period $T = 2\pi/\omega$ of the alternating electric field^{3, 4} one can obtain the solution $f^0(v)$ of the differential equation (1) in a straight-forward manner:

$$f^0(v) = C \exp \left[- \int_0^v \frac{m v' dv'}{k T + \frac{1}{6} M S(v') A_z^2} \right] \quad (4)$$

where

$$S(v') = \frac{2 \omega^2 [3 \nu^2 + \Omega^2 - \omega^2] + [\nu^2 + \Omega^2 - 3 \omega^2] [\nu^2 + (\omega_0 + \omega_B)^2 - \omega^2]}{\nu^2 [\nu^2 + \Omega^2 - 3 \omega^2]^2 + \omega^2 [3 \nu^2 + \Omega^2 - \omega^2]^2} \quad (5)$$

and

$$\Omega = \sqrt{(\omega_0 + \omega_B)^2 + \omega_c^2}, \quad \nu = \nu(v'). \quad (6)$$

The integration constant C can be determined by the normalization condition

$$4 \pi \int_0^\infty dv' v'^2 f^0(v') = n$$

n : electron number density.

2. Calculation of the Direction-Dependent Part f^1

The direction-dependent part $\mathbf{f}^1 = \{f_x^1, f_y^1, f_z^1\}$ of the of the electron distribution function can be found

with the help of the Table 1, 2, 3 of paper² making use of the same conditions which have been assumed for the calculation of f^0 . One gets

$$\begin{aligned} \mathbf{f}^1(v, t) &= \{ {}^+F_S \sin\{(\omega + \omega_B) t\} + {}^+F_C \cos\{(\omega + \omega_B) t\} \\ &\quad + {}^-F_S \sin\{(\omega - \omega_B) t\} + {}^-F_C \cos\{(\omega - \omega_B) t\}, \\ &\quad + {}^+G_S \sin\{(\omega + \omega_B) t\} + {}^+G_C \cos\{(\omega + \omega_B) t\} \\ &\quad + {}^-G_S \sin\{(\omega - \omega_B) t\} + {}^-G_C \cos\{(\omega - \omega_B) t\}, \\ &\quad \omega H_S \sin \omega t + \omega H_C \cos \omega t \} A_z \frac{\partial f^0}{\partial v} \\ &\equiv \{F, G, H\} A_z \frac{\partial f^0}{\partial v}. \end{aligned} \quad (7)$$

³ V. L. GINZBURG and A. V. GUREVIČ, Fortschr. Phys. **8**, 97 [1960].

⁴ E. A. DESLOGE and S. W. MATTHYSSE, Am. J. Phys. **28**, 1 [1960].

In Eq. (7) the coefficients ${}^{\omega}H_S$ and ${}^{\omega}H_C$ have the following meaning:

$$\begin{aligned} {}^{\omega}H_S &= \frac{1}{D} ({}^{\omega}a_{11} {}^{\omega}b_{11} + {}^{\omega}a_{12} {}^{\omega}b_{12}), \\ {}^{\omega}H_C &= \frac{1}{D} ({}^{\omega}a_{11} {}^{\omega}b_{12} - {}^{\omega}a_{12} {}^{\omega}b_{11}) \end{aligned} \quad (8)$$

where the abbreviations of Eq. (3) are used. The mathematical expressions of the coefficients ${}^+F_S$, ${}^+F_C$, ${}^-F_S$ etc. which are linear functions of ${}^{\omega}H_S$ and ${}^{\omega}H_C$ are listed in Table 1.

Knowing the parts f^0 and \mathbf{f}^1 of the velocity distribution function one easily finds the final form of the electron distribution function as a whole. According to the Lorentz ansatz and the Eqs. (4), (5), (6) the result is

$$\begin{aligned} f(v, t) &= f^0(v) + \frac{\mathbf{v}}{v} \cdot \mathbf{f}^1(v, t) \\ &= \{1 - (v_x F + v_y G + v_z H)\} \frac{m A_z}{k T + \frac{1}{2} M A_z^2} f^0(v) \end{aligned} \quad (9)$$

${}^+F_S$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu^2[\omega_0 + \omega_B + \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega]) + {}^{\omega}H_C(-\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 - 2\omega(\omega_0 + \omega_B)])]$
${}^+F_C$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 - 2\omega(\omega_0 + \omega_B)]) + {}^{\omega}H_C(\nu^2[\omega_0 + \omega_B + \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega])]$
${}^-F_C$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu^2[\omega_0 + \omega_B - \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B + \omega]) + {}^{\omega}H_C(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 + 2\omega(\omega_0 + \omega_B)])]$
${}^-F_S$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(-\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 + 2\omega(\omega_0 + \omega_B)]) + {}^{\omega}H_C(\nu^2[\omega_0 + \omega_B - \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega])]$
${}^+G_S$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 - 2\omega(\omega_0 + \omega_B)]) + {}^{\omega}H_C(\nu^2[\omega_0 + \omega_B + \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega])]$
${}^+G_C$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(-\nu^2[\omega_0 + \omega_B - \omega] - [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega]) + {}^{\omega}H_C(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 - 2\omega(\omega_0 + \omega_B)])]$
${}^-G_S$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 + 2\omega(\omega_0 + \omega_B)]) + {}^{\omega}H_C(-[\nu^2(\omega_0 + \omega_B - \omega)] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega])]$
${}^-G_C$	$\frac{\omega_C}{2N} [{}^{\omega}H_S(\nu^2[\omega_0 + \omega_B + \omega] + [(\omega_0 + \omega_B)^2 - \omega^2][\omega_0 + \omega_B - \omega]) + {}^{\omega}H_C(\nu[(\omega_0 + \omega_B)^2 + \nu^2 + \omega^2 + 2\omega(\omega_0 + \omega_B)])]$
$N = [\nu^2 + \omega^2 - (\omega_0 + \omega_B)^2]^2 + 4\nu^2(\omega_0 + \omega_B)^2$	

Table 1. Explanation of the coefficients ${}^+F_S$, ${}^+F_C$, ${}^-F_C$ etc.

where

$$\begin{aligned} F &= {}^+F_S \sin\{(\omega + \omega_B) t\} + {}^-F_C \cos\{(\omega + \omega_B) t\} \\ &\quad + {}^-F_S \sin\{(\omega - \omega_B) t\} + {}^-F_C \cos\{(\omega - \omega_B) t\}, \\ G &= {}^+G_S \sin\{(\omega + \omega_B) t\} + {}^+G_C \cos\{(\omega + \omega_B) t\} \\ &\quad + {}^-G_S \sin\{(\omega - \omega_B) t\} + {}^-G_C \cos\{(\omega - \omega_B) t\} \\ H &= {}^\omega H_S \sin \omega t + {}^\omega H_C \cos \omega t, \quad A_z = (q/m) |\mathbf{E}_z|. \end{aligned} \quad (10)$$

3. Generalized Cyclotron Resonance

The mathematical form of the isotropic part f^0 according to the external fields considered was given in the Eqs. (4) and (5). Now an identical expression for f^0 shall be derived permitting a more convenient physical interpretation. For this purpose let us introduce the variable β denoting the angle between the electric field vector \mathbf{E}_z in z direction and the vector of the effective magnetic field \mathbf{B}_{eff} (see Fig. 1). Using the abbreviations

$$\begin{aligned} \omega_0 &= (q/m) B_0, \quad \omega_C = (q/m) B_C, \quad v = v(v'), \\ \Omega &= (\omega_0^2 + (\omega_0 + \omega_B)^2)^{1/2} \end{aligned} \quad (11)$$

the expression $S(v')$ of Eq. (5) can be written in the form

$$\begin{aligned} S(v') &= \frac{\cos^2 \beta}{v^2(v') + \omega^2} \\ &\quad + \frac{\frac{1}{2} \sin^2 \beta}{v^2(v') + (\omega + \sqrt{\omega_0^2 + (\omega_0 + \omega_B)^2})^2} \\ &\quad + \frac{\frac{1}{2} \sin^2 \beta}{v^2(v') + (\omega - \sqrt{\omega_0^2 + (\omega_0 + \omega_B)^2})^2} \end{aligned} \quad (12)$$

The obvious physical meaning of the characteristic frequency Ω can be seen from Fig. 1. $\omega_B < \omega_C$ (a condition necessary for assumption (a) to be satisfied) and if no collisions are present every electron moves several times around the guiding centre which slowly rotates in the x, y -plane around the z axis.

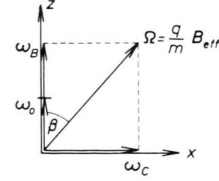


Fig. 1. Construction of \mathbf{B}_{eff} .

Because the constant magnetic field \mathbf{B}_0 and the rotating guiding centre (characterized by the fictive magnetic field $\mathbf{B}_f = (m/q) \boldsymbol{\omega}_B$) have the same action on the electrons the frequencies ω_0 and ω_B appear in the quantity Ω as the scalar sum $\omega_0 + \omega_B$, whereas the rotating magnetic field

$$\mathbf{B}^R = B_C \{ \cos \omega_B t, \sin \omega_B t, 0 \}$$

acts perpendicular to $\mathbf{B}_0 + \mathbf{B}_f$. Therefore, the effective magnetic field $\mathbf{B}_{\text{eff}} = (m/q) \boldsymbol{\Omega}$ is represented by the vector sum of $\mathbf{B}_0 + \mathbf{B}_f$ and \mathbf{B}^R .

Regarding the lowering of the breakdown fields near resonance conditions, results should be expected similar to those found by other investigators^{5, 6} for a plasma in a constant magnetic field alone.

If we perform a transition from the rotating magnetic field to a constant magnetic field (inclined to the electric field by an angle β) by putting the rotation frequency ω_B in $S(v')$ equal to zero we arrive at the FAİN-GUREVIČ distribution^{7, 8} with its well-known cyclotron resonance properties.

Finally, it is interesting to treat the case in which all magnetic fields disappear. Because the effective frequency Ω is equal to zero one gets

$$S(v') = 1/(v^2(v') + \omega^2)$$

that is, the isotropic part f^0 becomes a MARGENAU distribution⁹.

4. D. C. Field and Rotating Magnetic Field

Now the particular case is considered in which the circularly polarized magnetic field oscillates perpendicular to an electric d.c. field. The electron distribution function can be obtained by performing the limit $\omega \rightarrow 0$ in the preceding formulas.

Starting with the calculation of f^0 we get the quantity $S(v')$ from Eq. (12):

$$S(v') = \frac{\cos^2 \beta}{v^2(v')} + \frac{\sin^2 \beta}{v^2(v') + \Omega^2} = \frac{v^2(v') + (\omega_0 + \omega_B)^2}{v^2(v') (v^2(v') + \omega_C^2 + [\omega_0 + \omega_B]^2)}. \quad (13)$$

⁵ B. LAX, W. P. ALLIS, and S. C. BROWN, J. Appl. Phys. **21**, 1297 [1950].

⁶ V. AGNELLO, N. BARASSI, P. CALDIROLA et al., Nuovo Cim. **43**, 361 [1966].

⁷ B. M. FAİN, Zh. Eksperim. Teor. Fiz. **28**, 422 [1955].

⁸ A. V. GUREVIČ, Dokl. Akad. Nauk (UdSSR) **104**, 201 [1955].

⁹ H. MARGENAU, Phys. Rev. **69**, 508 [1946].

Ignoring a factor $1/2$ in the denominator of f^0 (because a time average is not necessary) we get the result

$$f^0(v) = C|_{\omega=0} \exp \left[- \int_0^v \frac{m v' dv'}{k T + \frac{M}{3} \frac{v^2(v') + (\omega_0 + \omega_B)^2}{v^2(v') + \omega_c^2 + (\omega_0 + \omega_B)^2} \cdot \frac{A_z^2}{v^2(v')}} \right]. \quad (14)$$

Some special cases of Eq. (14) are of interest:

a) Vanishing rotating magnetic field: $\omega_C = 0$. One gets

$$f^0(v) = C|_{\substack{\omega=0 \\ \omega_c=0}} \exp \left[- \int_0^v \frac{m v' dv'}{k T + \frac{M}{3} \frac{A_z^2}{v^2(v')}} \right]. \quad (15)$$

The result is a DRUYVESTEYN distribution³. The constant magnetic field (being parallel to the d.c. field) does not have any influence on f^0 .

b) Neglect of the rotation of the circularly polarized magnetic field: $\omega_B = 0$. The time-dependent magnetic field shrinks to $\mathbf{B}^R = \{B_C, 0, 0\}$, i. e. to a constant magnetic field in x direction. The plasma is then under the action of the d.c. field and the effective magnetic field $|\mathbf{B}_{eff}| = (B_0^2 + B_C^2)^{1/2}$. The result is

$$f^0(v) = C|_{\substack{\omega=0 \\ \omega_B=0}} \exp \left[- \int_0^v \frac{m v' dv'}{k T + \frac{M}{3} \frac{v^2(v') + \omega_0^2}{v^2(v') + \omega_c^2 + \omega_0^2} \cdot \frac{A_z^2}{v^2(v')}} \right]. \quad (16)$$

Putting in Eq. (16) the quantity ω_0 equal to zero we get a DAVYDOV distribution³.

c) Weak rotating magnetic field: $\omega_C \ll (v^2 + [\omega_0 + \omega_B]^2)^{1/2}$. In a good approximation the isotropic part f^0 is of DRUYVESTEYN's type, that is, both the time-dependent magnetic field \mathbf{B}^R and the constant magnetic field \mathbf{B}_0 do not influence the isotropic distribution.

For the final form of the part $\mathbf{f}^1(v, t)$ of the electron distribution function we get using Eqs. (7) and (8) and Table 1:

$$\mathbf{f}^1(v, t) = - \left\{ \frac{\omega_C [-v(v) \sin \omega_B t + (\omega_0 + \omega_B) \cos \omega_B t]}{v^2(v) + \omega_c^2 + (\omega_0 + \omega_B)^2}, \frac{\omega_C [(\omega_0 + \omega_B) \sin \omega_B t + v(v) \cos \omega_B t]}{v^2(v) + \omega_c^2 + (\omega_0 + \omega_B)^2}, \frac{v^2(v) + (\omega_0 + \omega_B)^2}{v^2(v) + \omega_c^2 + (\omega_0 + \omega_B)^2} \right\} \frac{A_z}{v(v)} \frac{\partial f^0}{\partial v}. \quad (17)$$

By use of the Lorentz ansatz (9) and the Eqs. (14) and (17) the total electron distribution function $f(\mathbf{v}, t)$ can be easily written down. A representation of the lengthy formula is omitted here¹⁰.

¹⁰ W. STILLER, Doctoral Thesis, Math.-Naturwiss. Fakultät, Karl-Marx-Universität Leipzig 1967.